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Solutions of the Diophantine Equation $A^4 + B^4 = C^4 + D^4$

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Abstract. A survey is presented of the more important solution methods of the equation of the title. When space permits, a brief description of the methods and numerical examples are also given. The paper concludes with an incomplete list of 218 primitive nontrivial solutions in rational integers not exceeding 10^6.

1. Introduction. The Diophantine equation

$$A^4 + B^4 = C^4 + D^4 \tag{1}$$

was first proposed by Euler [1] in 1772 and has since aroused the interest of numerous mathematicians. Among quartic Diophantine equations it has a distinct feature for its simple structure, the almost perfect symmetry between the variables and the close relationship with the theory of elliptic functions. The latter is demonstrated by the fact that Eq. (1) is satisfied by the four elliptic theta functions of Jacobi, ϑ_1, ϑ_3, ϑ_2 and ϑ_4, in that order [6].

One of the intriguing aspects of the equation is that numerical solutions are not easy to come by. Naturally, we are interested only in primitive and nontrivial solutions in real (and, occasionally, in Gaussian complex**) integers. The first known examples of solutions, and among these the solution in "least integers", i.e. $(A, B, C, D) = (134, 133, 158, 59)$, were computed already by Euler [1], [2], [3]. Some others were found by later researchers (see [4, pp. 644–647]), but it was not until the advent of computers that systematic searches could be conducted. The most extensive lists published to date are due to Lander and Parkin [16] and Lander, Parkin and Selfridge [17]. These lists, to be called LPS lists, contain 31 and 15 solutions, respectively, and are complete in their respective ranges.

In this paper we discuss the more important solution methods and in conclusion present a list of 218 numerical solutions. This contains all presently known primitive and nontrivial solutions in the range $\max(A, B, C, D) < 10^6$. Motivation to produce the list has come from the need for empirical material to study Eq. (1). To produce a largest possible selection of varied numerical examples, we have used all available methods at our disposal, while making no effort for completeness.

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** Results about solutions in Gaussian complex integers have been found by Ogilvy [15], Lander [18] and the present author, but are not included in this report.

2. Some Preliminary Remarks. Due to its special form, Eq. (1) is invariant under the transformations,

$$(2) \qquad S_A\colon A' = -A; \quad S_B\colon B' = -B; \quad S_C\colon C' = -C; \quad S_D\colon D' = -D,$$

$$(3) \qquad P_{AB}\colon A' = B, B' = A; \quad P_{CD}\colon C' = D, D' = C,$$

$$(4) \qquad P_{AC} \cdot P_{BD}\colon A' = C, B' = D, C' = A, D' = B,$$

and their products. We shall call these the *elementary transformations* of Eq. (1). Solutions obtained by elementary transformations from a given solution will not be considered different, but different forms of the same solution. Of the $2^7 = 128$ forms of any nontrivial solution we shall choose one as the *normal form* and define this by the following criteria:

(i) All four numbers A, B, C, D are positive.

(ii) B and D are odd.

(iii) The peak, i.e. $\max(A, B, C, D)$, is equal to B when it is odd and to C when it is even.

When there is no reason to do otherwise, the numerical solutions are quoted in their normal forms.

Following Euler, we shall use the notations $p = (C + A)/2$, $q = (C - A)/2$, $r = (B + D)/2$, $s = (B - D)/2$. With these substitutions we have

$$(5) \qquad pq(p^2 + q^2) = rs(r^2 + s^2),$$

an equation equivalent to (1). When computed from normal forms, all four numbers p, q, r, s are integers and positive.

3. Solution Methods. In contrast to the analogous cubic equation, no formula exists for the complete solution of Eq. (1). In its absence we have a large variety of methods at our disposal, each of which supplies a different set of solutions. The methods can be classified as (i) arithmetic methods, (ii) computer methods and (iii) mixed methods.

In the case of arithmetic methods we make special assumptions and use existing solutions to derive new ones. Since both the initial and derived solutions necessarily satisfy the same special conditions, no arithmetic method can yield all the solutions. However, it is possible to produce, at least in principle, a complete list of solutions in any given range by the application of computer methods.*** Naturally, in practice, the range of search is limited by the processing capacities of the computer used.

A pure computer method was used to produce the LPS lists. This is described in [16] and hence will not be discussed here.

In the case of mixed methods the computer search is coupled with an arithmetic preparation and subsequent algebraic calculations. In all known mixed methods the computer is used to check if a given algebraic expression takes the value of a perfect square. When this occurs, a solution of (1) is obtained by a further simple calculation.

*** As a matter of fact, at least for the present, this is possible only by computer methods. The fact that the solution $(A, B, C, D) = (134, 133, 158, 59)$ is the "smallest", was established also by the use of computer [14], and no other proof is known.

Of all the methods the simplified "Pythagorean triplets" method, a mixed method, has proved in practice the most efficient. The majority of solutions in the list was obtained by this method. We shall discuss it within the next section.

4. The Method of Pythagorean Triplets (PT). This method, in its original form, can be summed up as follows. Let (a_1, b_1, c_1) and (a_2, b_2, c_2) be Pythagorean triplets, i.e. numbers representable in the forms:

$$a_1 = 2u_1v_1, \quad b_1 = u_1^2 - v_1^2, \quad c_1 = u_1^2 + v_1^2$$

and

$$a_2 = 2u_2v_2, \quad b_2 = u_2^2 - v_2^2, \quad c_2 = u_2^2 + v_2^2$$

for some integers u_1, v_1, u_2, v_2. If the triplets are such that

$$(6) \qquad (a_1c_1 + a_2c_2)^2 + (b_1c_1 + b_2c_2)^2 = \text{perfect square},$$

then a solution of Eq. (1) is readily at hand. To obtain it, we first remove the common factor ρ of $\frac{1}{2} \cdot (a_1c_1 + a_2c_2)$ and $b_1c_1 + b_2c_2$ and then solve for U and V the system

$$(7) \qquad 2UV = \frac{1}{\rho} \cdot (a_1c_1 + a_2c_2), \qquad U^2 - V^2 = \frac{1}{\rho} \cdot (b_1c_1 + b_2c_2).$$

Then with

$$(8) \quad p = Uu_1 + Vv_1, \quad q = Uv_1 - Vu_1, \quad r = Uu_2 + Vv_2, \quad s = -Uv_2 + Vu_2$$

we have $A = p - q$, $B = r + s$, $C = p + q$, $D = r - s$ as solution of Eq. (1). Finally, we simplify by possible common factors of A, B, C, D and set them in normal form. The solution is nontrivial if the greatest common factor of $u_1^2 + v_1^2$, $u_2^2 + v_2^2$ and $u_1u_2 + v_1v_2$ equals 1. The formulas used in the method can easily be verified by applying (5).

When using this method, every solution will be obtained sooner or later. Moreover, it can be shown that every nontrivial solution can be computed from four different sets of parameters u_1, v_1, u_2, v_2, if their selection is subject to the restrictions (i) $u_1, v_1, u_2 > 0$, (ii) $u_1 > v_1, u_2, |v_2|$, (iii) the greatest common factor of $u_1^2 + v_1^2$, $u_2^2 + v_2^2$ and $u_1u_2 + v_1v_2$ is equal to 1. E.g. for the solution (134, 133, 158, 59) these sets are (26, 8, 14, 13), (45, 22, 6, –35), (55, 16, 40, –7) and (56, 34, 34, –31).

The disadvantage of the method lies in the difficulty of computing with high enough precision square roots of functional values of 8th degree polynomials. Nevertheless, when the method was first tried at the University of Zambia in 1972, 47 nontrivial solutions of (6) were obtained during one weekend night. Of these, 17 correspond to solutions not present in the LPS lists. The computer search was conducted by my former colleague, Jorma Pihlatie, using a relatively simple FORTRAN program and an IBM 1130 computer.

In a significant number of cases we have $v_1 = u_2$ or $u_1 = v_2$, and this observation has led to a modification of the method. For, if $v_1 = u_2$, then

$$a_1c_1 + a_2c_2 = 2x(u_1 + v_2)(u_1^2 - u_1v_2 + v_2^2 + x^2)$$

and

$$b_1c_1 + b_2c_2 = u_1^4 - v_2^4 = (u_1 + v_2)(u_1 - v_2)(u_1^2 + v_2^2),$$

where x denotes the common value $v_1 = u_2$. Hence $(u_1 + v_2)^2$ can be removed from the left-hand side of (6), which then reduces to

$$(9) \qquad 4x^2(u_1^2 - u_1v_2 + v_2^2 + x^2)^2 + (u_1 - v_2)^2(u_1^2 + v_2^2)^2 = \square.$$

This equation, or its simplified form,

$$(10) \qquad x^2(x^2 + 3y^2 + z^2)^2 + 4y^2(y^2 + z^2)^2 = \square,$$

(where $y = (u_1 - v_2)/2$ and $z = (u_1 + v_2)/2$), contains only 6th degree polynomials in three variables. Both the numerical work of polynomial evaluations and the dimension of search are hence reduced.

The majority of solutions marked by "PT" in the list was obtained by this simplified method. A search on the PDP-10 computer of the State University of Campinas has produced 222 nontrivial primitive solutions of (9) in the range $u_1, v_1 \leq 1061$, $u_1 > |v_2|$. However, not all corresponding solutions of Eq. (1) are contained in the list. Excluded are 26 solutions whose peaks exceed 10^6. Further, there are many instances of coincidence, i.e. different solutions of (9) leading to the same solution of (1). (Every nontrivial solution of Eq. (1) can be obtained from 8 different primitive sets of parameters.) Hence the number of solutions marked by "PT" falls well below 222.

5. Semisolution Methods. By a *semisolution* of Eq. (1) we mean a parametric solution $A = A(u, v, t)$, $B = B(u, v, t)$, $C = C(u, v, t)$, $D = D(u, v, t)$, where the parameters u, v and t have to satisfy a further Diophantine equation $Q(u, v) = t^2$. Here $Q(u, v)$ denotes a homogeneous quartic polynomial. Through the semisolutions the problem of solving Eq. (1) is thus reduced to the problem of making a quartic a perfect square.

Quite frequently, in lieu of A, B, C, D, the numbers p, q, r, s are given as functions of u, v, t, as e.g. in the semisolution

$$(11) \qquad p = ft, \quad q = gu(f^2u - g^2v), \quad r = gt, \quad s = fv(f^2u - g^2v),$$

which goes back to Euler [2]. Here f and g are integral constants (free parameters) and

$$(12) \qquad t^2 = (f^2u - g^2v)(f^2v^3 - g^2u^3).$$

The equation $Q(u, v) = t^2$ will be referred to as the *quartic equation* of the semisolution in question. We shall agree that a solution of a quartic equation is termed nontrivial, if it leads to a nontrivial solution of Eq. (1).

The importance of solving quartic equations was recognized already by Euler who himself gave three different methods to make a quartic a perfect square. Following Euler several other methods have become known, but the basic problem, namely to find the *complete* solution of *any* given quartic equation has still remained unsolved. A method most frequently used is the following. Suppose we have found a representation of the quartic $Q(u, v)$ in discriminant form, i.e.

$$(13) \qquad Q(u, v) = \beta^2(u, v) - \alpha(u, v) \cdot \gamma(u, v),$$

where $\alpha(u, v)$, $\beta(u, v)$ and $\gamma(u, v)$ denote quadratic homogeneous polynomials in u and v. Suppose further that a solution u_0, v_0, t_0 of the quartic equation is already known. Then the roots of the quadratic equation

$$\alpha(u_0, v_0) \cdot x^2 - 2\beta(u_0, v_0) \cdot xy + \gamma(u_0, v_0) \cdot y^2 = 0$$

are rational numbers, namely

$$\frac{x_0}{y_0} = \frac{\beta(u_0, v_0) - t_0}{\alpha(u_0, v_0)} \quad \text{and} \quad \frac{x_1}{y_1} = \frac{\beta(u_0, v_0) + t_0}{\alpha(u_0, v_0)}.$$

Now the equation

$$\alpha(u, v) \cdot x^2 - 2\beta(u, v) \cdot xy + \gamma(u, v) \cdot y^2 = 0 \tag{14}$$

is quadratic and homogeneous in u and v. Moreover, when we put $x/y = x_n/y_n$, $n = 0, 1$ it has one rational solution for u/v, namely u_0/v_0. It follows that the other solution must also be rational. In this way we obtain solutions u_{-1}/v_{-1} (when $x/y = x_0/y_0$) and u_1/v_1 (when $x/y = x_1/y_1$). Repeating this argument with the new values u_{-1}, v_{-1} and u_1, v_1, we obtain further solutions, etc. The ratios u_n/v_n and x_n/y_n form, in general, a doubly infinite chain,†

$$\ldots, \frac{u_{-1}}{v_{-1}}, \frac{x_0}{y_0}, \frac{u_0}{v_0}, \frac{x_1}{y_1}, \frac{u_1}{v_1}, \frac{x_2}{y_2}, \ldots, \tag{15}$$

determined by the equation (14) and an initial ratio u_0/v_0. Accordingly, we shall call Eq. (14) a *chain-generating equation*.

For a detailed account of the various arithmetic methods see Dickson [4, pp. 639–644]. All these are equivalent, in one way or another, to the chord and tangent process of finding rational points on an elliptic curve (see [8, Chapter 16]). It is a well-known fact that, by applying this process, all rational points can be generated from a finite set of them. Consequently, all solutions of a quartic Diophantine equation, $Q(u, v) = t^2$, can be found from a finite set of solutions by arithmetic methods. The main difficulty is that no known method exists to determine this finite set in the general case. Otherwise it would be possible to determine it e.g. for the quartic equation

$$u^4 - Mv^4 = t^2$$

with general integral M. For this equation it is known that when M is representable in the form $a^4 + b^4$, it has the independent solutions

$$u = a^2 + ab + b^2, \quad v = a + b, \quad t = ab \cdot (2a^2 + 3ab + 2b^2)$$

and

$$u = a^2 - ab + b^2, \quad v = a - b, \quad t = ab \cdot (2a^2 - 3ab + 2b^2)$$

When M is representable as a sum of two biquadrates in two different ways, we have two more solutions of this kind. The number of solutions in the finite set hence depends, among others, on the number of ways of representing M in the form $a^4 + b^4$. Thus the problem goes back to solving Eq. (1).

† The other possibility is a cycle.

The possible failure of arithmetic methods notwithstanding, computers can always be used within the limits of their capacities to solve quartic equations. Then the result is a complete list of solutions in the range of search. When the quartic equation of a semisolution is solved this way, we have another instance of solution methods of mixed type.

6. Some Examples of Semisolutions. The second example of semisolutions that appeared in the literature, was also given by Euler [3]. In simplified formulation we quote it as follows:

$$\begin{aligned} A &= 2Pu^2 + Qv^2, \\ SB &= t, \\ C &= 2Pu^2 + 2(Q-P)uv - Qv^2, \\ SD &= -2P(Q-P)u^2 + 4PQuv + Q(Q-P)v^2, \end{aligned}$$

with the quartic equation,

$$\begin{aligned} t^2 = {} & 4P^2(Q-P)^2u^4 + 8P(Q-P)(Q^2+P^2)u^3v \\ & +4(Q^4 - 3Q^3P - 3QP^3 + P^4)u^2v^2 \\ & -4Q(Q-P)(Q^2+P^2)uv^3 + Q^2(Q-P)^2v^4. \end{aligned}$$

The parameters P, S, Q form a Pythagorean triplet, i.e. $P^2 + S^2 = Q^2$, but are otherwise unspecified. We have nontrivial solutions when $Q^2 + QP + P^2$ is a perfect square, as in the case $P = 3$, $Q = 5$, observed by Euler and leading to Solution 5, (see also [9]). However, these are not the only nontrivial solutions. An example when $Q^2 + QP + P^2$ is not a perfect square is the following: $P = 400$, $Q = 689$, $S = 561$. Then we have the solution $u = 51$, $v = 20$, $t = 761210360$, leading to Solution 43 of the list.

Strictly speaking, the simplified PT method is also a semisolution method, since Eq. (10) turns into a quartic equation by substituting u/v for z and multiplying every term by v^4. Another semisolution can be derived from the original PT method by assuming that $v_2 = 0$. Then we have $a_2 = 0$, $b_2 = c_2 = u_2^2$, and hence by (7)

$$2UV = \frac{1}{\rho} \cdot 2u_1v_1(u_1^2 + v_1^2) \quad \text{and} \quad U^2 - V^2 = \frac{1}{\rho} \cdot (u_1^4 - v_1^4 + u_2^4).$$

Without loss of generality we may set $U = \kappa(u_1^2 + v_1^2)$, $V = u_1v_1/(\kappa\rho)$, with κ denoting an appropriate constant. Substituting these in the second equation, we have

$$\kappa^2(u_1^2 + v_1^2)^2 - \frac{1}{\kappa^2\rho^2} \cdot u_1^2v_1^2 = \frac{1}{\rho} \cdot (u_1^4 - v_1^4 + u_2^4).$$

In the simplest case, i.e. when $\kappa^2\rho = 1$, this last equation reduces to

$$u_1^2v_1^2 = u_2^4 - 2v_1^4.$$

Now the equation

$$t^2 = u^4 - 2v^4 \tag{16}$$

is known to have infinitely many nontrivial solutions (see [8, pp. 72–74]). Using these, we have the following expressions for u_1v_1, u_2 and v_1: $u_1v_1 = \sigma^2 t$, $u_2 = \sigma u$, $v_1 = \sigma v$, whence, by choosing $\sigma = v$, $u_1 = t$, $v_1 = v^2$ and $u_2 = uv$ follow. For

p, q, r, s then we have, by (8),

$$p = u^4 t, \quad q = v^6, \quad r = uv(u^4 - v^4), \quad s = uv^3 t. \tag{17}$$

Applying the simplest nontrivial solution of (16), i.e. $(u, v, t) = (3, 2, 7)$, the result is Solution 6 of the list.

Finally, let us mention the semisolution of Fauquembergue [10], who gave it as an identity. We present it in the following formulation:

$$\begin{aligned} A &= t, \\ B &= 4u^4 + 9uv + 4v^2, \\ C &= 4u^2 + 15uv - 2v^2, \\ D &= -2u^2 + 15uv + 4v^2, \end{aligned} \tag{18}$$

with the quartic equation,

$$t^2 = 4u^4 + 132u^3v + 17u^2v^2 + 132uv^3 + 4v^4. \tag{19}$$

Fauquembergue's example can be easily generalized and developed into a full-scale theory. In the next section, however, we shall give only the main results, owing to the considerable length of calculations.

7. Fauquembergue Type Semisolutions. Observing that a sum of two biquadrates in two ways is also a sum of two squares in two ways and that as such it can be represented as a product of two sums of two squares each, we set

$$A^4 + B^4 = C^4 + D^4 = (a^2 + b^2)(c^2 + d^2)$$

and choose

$$A^2 = ac - bd, \quad B^2 = ad + bc, \quad C^2 = ac + bd, \quad D^2 = ad - bc. \tag{20}$$

Then

$$C^2 - B^2 = (a - b)(c - d) \quad \text{and} \quad B^2 - D^2 = 2bc.$$

Hence, without loss of generality, we may assume that

$$C + B = \mu(a - b), \quad C - B = \frac{1}{\mu}(c - d), \quad B + D = 2\nu c, \quad B - D = \frac{b}{\nu} \tag{21}$$

for some μ and ν. We shall denote the product $\mu\nu$ by τ. This quantity, which occurs frequently in the formulas, playing the role of an invariant, has the following expressions, derivable from (21) and (20),

$$\tau = \frac{B + C}{a - b} \cdot \frac{B + D}{2c} = \frac{(B + C)(B + D)}{A^2 - B^2 + C^2 + D^2}, \tag{22.a}$$

and

$$\tau = \frac{c - d}{C - B} \cdot \frac{b}{B - D} = \frac{A^2 + B^2 - C^2 - D^2}{2(C - B)(B - D)}. \tag{22.b}$$

For $2B$ we obtain from (21) the expressions,

$$\mu(a - b) - \frac{1}{\mu} \cdot (c - d) \quad \text{and} \quad \frac{b}{\nu} + 2\nu c,$$

which, when equated, yield a linear relation between the four parameters a, b, c, d. In addition there exists a quadratic relation, too, namely

$$4ad = 2B^2 + 2D^2 = (B + D)^2 + (B - D)^2 = (2\nu c)^2 + \left(\frac{b}{\nu}\right)^2. \tag{23}$$

Using the linear relation, we can reduce this last one, (23), to an equation in 3 variables. Introducing 3 new variables, λ, ε and φ, defined by the linear substitutions,

$$\lambda = d - a\mu^2, \tag{24}$$

$$\varepsilon = \frac{b\mu}{\nu} = \frac{d + a\mu^2 - (2\tau + 1)c}{\tau + 1}, \tag{25}$$

$$\varphi = (4\tau + 1)c + (\tau + 1)(2\tau + 1)\varepsilon, \tag{26}$$

the result is a Diophantine equation in which only pure quadratic terms appear, namely

$$\varphi^2 - (4\tau + 1)\lambda^2 = \Delta_1\Delta_2\varepsilon^2. \tag{27}$$

Here we use, for brevity, the notations

$$\Delta_1 = 2\tau^2 + 1, \qquad \Delta_2 = 2\tau^2 + 4\tau + 1. \tag{28}$$

The solution of (27) is straightforward (see, e.g., [5, Chapter 4, Section 29]). Using a known solution, φ_0, λ_0, ε_0, and two free parameters, u and v, the complete solution of (27) may be written as follows:

$$\begin{aligned} \rho\varphi &= \varphi_0 u^2 + 2\lambda_0(4\tau + 1)uv + \varphi_0 \cdot (4\tau + 1)v^2, \\ \rho\lambda &= \lambda_0 u^2 + 2\varphi_0 uv + \lambda_0 \cdot (4\tau + 1)v^2, \\ \rho\varepsilon &= \varepsilon_0 u^2 - \varepsilon_0 \cdot (4\tau + 1)v^2, \end{aligned} \tag{29}$$

with ρ denoting a proportionality factor. This can be dropped (or its value set to be equal to 1) since we are interested only in the ratios $\varphi : \lambda : \varepsilon$. The initial solution, φ_0, λ_0, ε_0, is returned by the choice $u = 1, v = 0$.

An initial solution φ_0, λ_0, ε_0 of (27) is readily available from a known solution A_0, B_0, C_0, D_0 of Eq. (1), using the linear relations (21), (24), (25) and (26). As a result, the complete solution of (27) can be expressed in the terms of B_0, C_0, D_0 instead of φ_0, λ_0, ε_0, and so can the parameters a, b, c, d and the variables B, C, D. The formulas for the latter are quoted as follows:

$$\begin{aligned} B = B_0u^2 &- \left[(2\tau^2 - 1)B_0 + 4\tau C_0 - (2\tau^2 + 1)D_0\right]uv \\ &+ \left[(4\tau^3 + 6\tau^2 + 2\tau)B_0 - (4\tau^3 + 6\tau^2 - 2\tau - 1)D_0\right]v^2, \end{aligned} \tag{30}$$

$$\begin{aligned} C = C_0u^2 &- \left[(2\tau^2 + 4\tau + 3)B_0 + 2C_0 - (2\tau^2 + 4\tau - 1)D_0\right]uv \\ &+ \left[(6\tau^2 + 4\tau + 1)(B_0 - D_0) + (4\tau + 1)C_0\right]v^2, \end{aligned} \tag{31}$$

$$\begin{aligned} D = D_0u^2 &- \left[(2\tau^2 - 1)B_0 + 4\tau C_0 - (2\tau^2 + 1)D_0\right]uv \\ &+ \left[(4\tau^3 + 6\tau^2 + 6\tau + 1)B_0 - (4\tau^3 + 6\tau^2 + 2\tau)D_0\right]v^2. \end{aligned} \tag{32}$$

These three formulas, together with

$$A = t, \tag{33}$$

express a semisolution whose quartic equation is

$$(34)\qquad t^2 = C^2 + D^2 - B^2 + 2\tau(C - B)(B - D).$$

Here on the right-hand side, the expressions given at (30)–(32) are to be substituted for B, C and D. The relation (34) is an immediate consequence of (22.b) and (33).

The semisolution just derived has a structure similar to the one of Fauquembergue's example (18)–(19). We can obtain infinitely many others from it by subjecting the parameters u, v to linear (nonsingular) transformations. We shall refer to all these as *Fauquembergue type semisolutions*, or briefly F-*solutions*.

The quartic in (34) can be brought into a discriminant form (13) in various ways and thus be solved arithmetically. Following are some discriminant forms of which the last two are symmetric in C and D.

$$(35)\qquad t^2 = C^2 - (B - D)[B + D + 2\tau(B - C)],$$

$$(36)\qquad t^2 = (B - C - D)^2 - 2(\tau + 1)(B - C)(B - D),$$

$$(37)\qquad t^2 = [(2\tau + 1)B + C + D]^2 - 4(\tau + 1)[(\tau + 1)B + C - \tau D][(\tau + 1)B + D - \tau C].$$

A further one can be obtained from (35) by interchanging C and D.

Numerical Example. Starting from $(A_0, B_0, C_0, D_0) = (292, 193, 256, 257)$, (a form of Solution 3), we have by (22.a)

$$\tau = \frac{(B_0 + C_0)(B_0 + D_0)}{A_0^2 - B_0^2 + C_0^2 + D_0^2} = \frac{449 \cdot 450}{179600} = \frac{9}{8}.$$

Using this value, we can now compute the right-hand sides of (30)–(32) and (34). The result is the semisolution

$$(38)\qquad \begin{aligned} A &= t, \\ B &= 193u^2 - 540uv + 419v^2, \\ C &= 256u^2 - 898uv + 570v^2, \\ D &= 257u^2 - 540uv + 67v^2, \end{aligned}$$

with the quartic equation,

$$(39)\qquad t^2 = 85264u^4 - 477344u^3v + 999100u^2v^2 - 927096uv^3 + 273420v^4.$$

The discriminant form (37) of the quartic is as follows:

$$t^2 = \frac{1}{16} \cdot (4561u^2 - 12772uv + 7995v^2)^2 - \frac{153}{16} \cdot (377u^2 - 1438uv + 1385v^2)(337u^2 - 602uv + 281v^2).$$

Hence we may set

$$\alpha(u, v) = 377u^2 - 1438uv + 1385v^2,$$
$$\beta(u, v) = 4561u^2 - 12772uv + 7995v^2$$

and

$$\gamma(u, v) = 153(337u^2 - 602uv + 281v^2)$$

as coefficients of the chain-generating equation, (14), and compute elements of the chain. The initial values $u_0 = 1$, $v_0 = 0$ (that correspond to the initial solution A_0, B_0, C_0, D_0) give for x_0/y_0 the ratio 9/1, and for x_1/y_1 the ratio 5729/377. Using $x_0 = 9$, $y_0 = 1$, we obtain $u_{-1}/v_{-1} = -313/592$. On substituting 313 for u and -592 for v in (38) and (39), we obtain Solution 41, i.e. (12772, 9153, 13472, 5121), after removing the common factor 29041.

From the formulas (30)–(34) it is clear that F-solutions can be derived from every (nontrivial) numerical solution of Eq. (1). Moreover, since we have 16 different τ-invariants for every nontrivial solution of Eq. (1) (these are obtained from formulas (22.a) or (22.b) by applying the elementary transformations to A, B, C, D), it is easily seen that every nontrivial solution of Eq. (1) generates 16, essentially different, F-solutions.

In the special case when $4\tau + 1$ is equal to a rational square, say $(2n+1)^2$, i.e. $\tau = n(n+1)$ for some rational n, the complete solution of (27) can be expressed without the use of a known solution, namely as

$$(40)\quad \rho\varphi = (2n+1)\cdot\left(\Delta_1 u^2 + \Delta_2 v^2\right),\quad \rho\lambda = \Delta_1 u^2 - \Delta_2 v^2,\quad \rho\varepsilon = 2(2n+1)\cdot uv.$$

Using these, we can derive the following semisolution:

$$(41)\quad \begin{aligned} A &= t,\\ B &= \tau\Delta_1 u^2 - (4\tau^3 + 6\tau^2 - 2\tau - 1)uv + \tau\Delta_2 v^2,\\ C &= -n\Delta_1 u^2 - (6\tau^2 + 4\tau + 1)uv + (n+1)\Delta_2 v^2,\\ D &= \tau\Delta_1 u^2 - (4\tau^3 + 6\tau^2 + 6\tau + 1)uv + \tau\Delta_2 v^2, \end{aligned}$$

with the quartic equation,

$$(42)\quad \begin{aligned} t^2 = {} & n^2\Delta_1^2 u^4 - 2n\Delta_1\left[2\tau^2 - 2\tau - 1 + 2(n+1)(\tau+1)(4\tau+1)\right]u^3 v\\ & + \left[(2\tau+1)\Delta_1\Delta_2 + 8\tau^2(\tau+1)(6\tau^2+4\tau+1)\right]u^2v^2\\ & + 2(n+1)\Delta_2\left[2\tau^2 - 2\tau - 1 - 2n(\tau+1)(4\tau+1)\right]uv^3\\ & + (n+1)^2\Delta_2^2 v^4. \end{aligned}$$

Formulas (41) and (42) become trivial and hence useless when $n = 0$, -1 or $-1/2$. Otherwise n may take any rational value. When $n = -1/2$, (41) and (42) are replaced by the following:

$$(43)\quad \begin{cases} A = t,\\ B = 2u^2 - uv - 2v^2,\\ C = 2u^2 - uv + 4v^2,\\ D = -4u^2 - uv - 2v^2, \end{cases}$$

$$(44)\quad t^2 = 16u^4 + 8u^3v + 23u^2v^2 - 8uv^3 + 16v^4.$$

Fauquembergue's example (18)–(19) corresponds to the case $\tau = 2$, i.e. $n = 1$ or -2, and can be obtained from (41)–(42) by an appropriate linear transform on u and v.

The quartic in (42) has—among others—the following discriminant form:

$$(45)\quad \begin{aligned} t^2 = {} & \left[n\Delta_1 u^2 - \left(2\tau^2 - 2\tau - 1 + 2(n+1)(\tau+1)(4\tau+1)\right)uv\right.\\ & \qquad \left. + (n+1)(6\tau^2 + 6\tau + 1)v^2\right]^2\\ & - 4\tau(\tau+1)(2\tau+1)(4\tau+1)\left[(n+2)u - (n+1)v\right]^2 v^2. \end{aligned}$$

When using this form to generate a chain (15), the resulting new solutions of Eq. (1) are in general different from those obtainable through the use of the other discriminant forms (35)–(37).

8. Algebraic Reductions. By making special assumptions, Eq. (1) can be reduced to a linear one and thus solved promptly. One way to do this is by applying *Cauchy's method* of reducing cubic homogeneous Diophantine equations in three unknowns [7]. For this we start from Eq. (5), which is already of 3rd degree in each of its four variables, and observe with Desboves [11] that it can be made a 3-variable homogeneous equation, e.g. by assuming that $p/r = \mu =$ const and eliminating p. However, the same results can be obtained more quickly by direct methods which are possible due to the special symmetric character of Eq. (1). Depending on the assumptions to be made, we arrive at the methods (i) of *Lander* [18], (ii) of *Swinnerton-Dyer* [13], (iii) *the perfect cube method* and (iv) *the two-solution method.*

In the case of *Lander's method* we add to the assumption

$$p/r = (C + A)/(B + D) = \mu = \text{const a similar one,}$$

namely $(A + D)/(C - B) = \nu =$ const. Then, by denoting the variables of the new solution by primes, we have

$$p' = px, \qquad r' = rx, \tag{46}$$

$$A' + D' = (A + D)y, \qquad C' - B' = (C - B)y, \tag{47}$$

i.e.

$$q' = qy + (x - y)r, \qquad s' = sy + (x - y)p. \tag{48}$$

Substituting the expressions for p', q', r' and s' in the equation

$$p'q'(p'^2 + q'^2) - r's'(r'^2 + s'^2) = 0, \tag{49}$$

it is readily seen that this reduces to a linear equation in x and y, since it can be simplified by $xy(x - y)$ for obvious reasons. After simplifications we obtain

$$x(rs^3 - pq^3 + 2rp^3 - 2pr^3 + 3pqr^2 - 3rsp^2) + y\left[r(s - p)^3 - p(q - r)^3\right] = 0,$$

Thus

$$\rho x = p(q - r)^3 - r(s - p)^3 \tag{50}$$

and

$$\rho y = rs^3 - pq^3 + 2rp^3 - 2pr^3 + 3pqr^2 - 3rsp^2, \tag{51}$$

where ρ is an appropriate proportionality factor. Substituting (50) and (51) in (46) and (48), we obtain p', q', r', s' as 5th degree functions of p, q, r, s.

The relationship between the original and the new solutions is symmetric as is evident from the formulas (46) and (47). That means that the initial variables p, q, r, s, are also 5th degree functions of p', q', r', s'. (For this reason the author terms the method as a *dual transform* of the variables.) It is also evident that by changing the signs of A, B, C, D, each of the eight, essentially different, sign-combinations yield a different solution. Thus any nontrivial solution leads to 8 new solutions. Each of these 8 solutions can of course be employed again to obtain 8 other solutions and so on up to infinity. However, among the 8 solutions obtained from any of the first eight there are only seven new ones, the other one being identical to the initial solution for the symmetry mentioned above.

To illustrate this point, in Tables 1 and 2, respectively, we list the 8 solutions, in normal forms and in increasing order of their peaks, that are obtained from the solutions 2 and 3, i.e. (7, 239, 227, 157) and (256, 257, 292, 193), respectively. These examples serve also to show how widely the order of magnitude of the new solutions varies. Other examples were given in [18] and [19].

TABLE 1

No.	*A*	*B*	*C*	*D*
1	256	257	292	193
2	3 364	4 849	4 288	4 303
3	94 108	378 507	333 384	301 387
4	219 380	858 201	840 360	463 207
5	840 766	518 255	869 338	161 105
6	1 247 062	1 221 659	1 466 462	381 787
7	154 215 814	112 532 691	164 145 966	6 129 427
8	480 321 046	695 642 811	732 188 802	20 030 203

TABLE 2

No.	*A*	*B*	*C*	*D*
1	7	239	227	157
2	248	2 797	2 524	2 131
3	3 080	39 789	30 348	35 885
4	21 708	1 102 237	1 047 672	721 699
5	732 965	11 610 623	11 589 385	3 395 261
6	4 925 561	37 899 133	37 834 817	10 984 277
7	524 937 467	3 830 530 437	3 823 811 431	1 121 601 087
8	7 493 624 732	2 184 895 107	7 507 106 424	432 984 899

In the case of *Swinnerton-Dyer's method* we keep the transformation formulas (46), but replace (48) by the following ones:

$$q' = qx + vy, \qquad s' = sx + uy. \tag{52}$$

Substituting these in (49) and simplifying by x and y, we obtain a quadratic equation in x and y, i.e.

$$(p^3v + 3pq^2v - r^3u - 3rs^2u)x^2 + 3(pqv^2 - rsu^2)xy + (pv^3 - ru^3)y^2 = 0. \tag{53}$$

This can further be reduced by giving u and v values that make the coefficient of x^2 zero, namely

$$u = p(p^2 + 3q^2), \qquad v = r(r^2 + 3s^2). \tag{54}$$

Thus, from (53),

$$y/x = -3(pqv^2 - rsu^2)/(pv^3 - ru^3),$$

whence, after substitutions and simplifications,

$$(55)\qquad \begin{aligned} \rho x &= r^2(r^2+3s^2)^3 - p^2(p^2+3q^2)^3, \\ \rho y &= -3qr(r^2+3s^2)^2 - ps(p^2+3q^2)^2, \end{aligned}$$

with ρ denoting a proportionality factor.

Substituting the values (55) and (54) in (52), we obtain the variables of the new solution as 9th degree functions of the variables of the initial solution. Unlike Lander's method, here the relationship between the original and the new solution is not symmetric. However, the number of derived solutions obtainable from a nontrivial solution is also eight, since different sign-combinations and/or different orders of A, B, C, D (see Elementary transformations, Section 2) lead to different solutions.

Despite this prolific character, the method is of little use in practice, unless we do not mind obtaining solutions in big numbers. E.g. starting from $(A, B, C, D) = (-7, 239, 227, 157)$, (a form of Solution 2), we obtain

$$A' = 96\,781\,561\,849, \qquad B' = -22\,312\,231\,691,$$
$$C' = -57\,072\,919\,679, \qquad D' = 93\,787\,787\,597$$

as new solution. The sudden increase in the order of magnitude of new solutions is due to the 9th degree expressions mentioned above.

The *perfect cube method* can be applied only in the special case when for the initial solution the ratio p/r is the cube of a rational number, u/v. Then the coefficient of y^2 in (53) disappears and thus

$$x/y = -3(pqv^2 - rsu^2)/\left[pv(p^2+3q^2) - ru(r^2+3s^2)\right],$$

whence, after simplifications,

$$(56)\qquad \rho x = -3uv(uq - vs), \qquad \rho y = u^2(p^2+3q^2) - v^2(r^2+3s^2).$$

Substituting x and y in (46) and (52), we obtain a new solution.

As an example, let us start from the semisolution,

$$(p, q, r, s) = (u^4t, v^6, uv^3t, uv(u^4 - v^4)),$$

where $t^2 = u^4 - 2v^4$ (a variant of semisolution (17)). Since $p/r = u^3/v^3$, the method can be applied. Thus, by (56), $\rho x = 3u^2v^3t^2$ and $\rho y = u^2t^2(u^8 - 3u^4v^4 - v^8)$. Letting $\rho = u^2t^3$, from (46) and (52) we obtain for the new variables:

$$(57)\qquad p' = 3u^4v^3, \quad q' = vt(u^4 - v^4), \quad r' = 3uv^6, \quad s' = ut(u^4 + 2v^4).$$

This new solution represents, of course, also a semisolution with the same quartic equation. Substituting $(u, v, t) = (3, 2, 7)$, the result is Solution 12.

We wish to remark that Euler's well-known 7th degree parametric solution can also be deduced this way. For the initial solution we choose the trivial solution $(p, q, r, s) = (u^3, v^3, v^3, u^3)$ and have, by (56),

$$\rho x = 3u^2v^2(u^2 - v^2), \qquad \rho y = (u^2 - v^2)(u^6 - 2u^4v^2 - 2u^2v^4 + v^6).$$

Letting $\rho = u^2 - v^2$, from (46) and (52) we obtain

$$p' = 3u^5v^2, \qquad q' = v(u^6 - 2u^4v^2 + u^2v^4 + v^6),$$
$$r' = 3u^2v^5, \qquad s' = u(u^6 + u^4v^2 - 2u^2v^4 + v^6),$$

which is a variant of Euler's solution.

Finally we discuss the *two-solution method*. We denote the known solutions of (5) by (p_1, q_1, r_1, s_1) and (p_2, q_2, r_2, s_2), and assume that

$$p_1/r_1 = p_2/r_2 = \mu = \text{const.} \tag{58}$$

The variables of the new solution are set as

$$p' = p_1x + p_2y, \quad q' = q_1x + q_2y, \quad r' = r_1x + r_2y, \quad s' = s_1x + s_2y. \tag{59}$$

Since by (58) $p'/r' = p_1/r_1 = p_2/r_2$, the equation (49) reduces to a cubic equation in x and y, namely

$$p_1(q_1x + q_2y)\left[(p_1x + p_2y)^2 + (q_1x + q_2y)^2\right] - r_1(s_1x + s_2y)\left[(r_1x + r_2y)^2 + (s_1x + s_2y)^2\right] = 0.$$

In this equation the coefficients of x^3 and y^3 are equal to zero (because (p_1, q_1, r_1, s_1) and (p_2, q_2, r_2, s_2) satisfy (5)), and the ratio x/y can thus be readily computed:

$$\begin{aligned} \rho x &= p_1\left(q_1p_2^2 + 2p_1p_2q_2 + 3q_1q_2^2\right) - r_1\left(s_1r_2^2 + 2r_1r_2s_2 + 3s_1s_2^2\right), \\ \rho y &= -p_1\left(q_2p_1^2 + 2p_1p_2q_1 + 3q_1^2q_2\right) + r_1\left(s_2r_1^2 + 2r_1r_2s_1 + 3s_1^2s_2\right). \end{aligned} \tag{60}$$

Substituting x and y in (59), the variables of the new solution are obtained as 5th degree functions of the initial variables. The relationship between the two initial solutions and the derived solution, however, is quite symmetric, as is evident from (59). That means that by choosing e.g. (p_1, q_1, r_1, s_1) and (p', q', r', s') as initial solutions, (p_2, q_2, r_2, s_2) is obtained as a derived solution.

It should further be noted that from any two solutions satisfying (58) there can actually be obtained one more solution by this method. For this we change the signs of p_2 and r_2 in (59) and (60).

As a numerical example, we mention the computation of Solution 98 by this method, starting from

$$(p_1, q_1, r_1, s_1) = (4563, 1409, 845, 5535)$$

(obtained from a form of Solution 13) and

$$(p_2, q_2, r_2, s_2) = (-1053, 1714, -195, 3342)$$

(obtained from a form of Solution 15), for which $p_1/r_1 = p_2/r_2 = 27/5$. The other solution, computed by using

$$(p_2, q_2, r_2, s_2) = (1053, 1714, 195, 3342),$$

is the following:

$$(A, B, C, D) = (13721986, 37753977, 34224263, -28875042).$$

The two-solution method includes, as a special case, the method of Lander. To show this, we choose for initial solutions a known solution, $(p_1, q_1, r_1, s_1) = (p, q, r, s)$, and an associated trivial solution, $(p_2, q_2, r_2, s_2) = (p, r, r, p)$. Substituting these in (60), for the ratio $x/(x + y)$ we have

$$\left(rs^3 - pq^3 + 2rp^3 - 2pr^3 + 3pqr^2 - 3rsp^2\right)/\left[p(q-r)^3 - r(s-p)^3\right].$$

Hence, in view of (50) and (51), we are back at the method of Lander. The two-solution method can thus be regarded as a generalization of Lander's method.

In all four methods discussed in this section we have $p/r =$ const, and this same condition is satisfied also by the semisolution (11)–(12). In fact, all derived solutions can also be obtained from the semisolution formulas by using different techniques to solve the quartic equation (12). For instance, when we use the discriminant-form,

$$(61)\quad \left[g^2u^2 + (f^2 + g^2)uv + f^2v^2\right]^2 \\ -\left[g^2(u^2 + uv + v^2) + f^2v^2\right]\left[g^2u^2 + f^2(u^2 + uv + v^2)\right],$$

of the quartic in (12) to set up a chain-generating equation, the resulting solutions of Eq. (1) are identical to those obtainable by Lander's method.

9. Parameter Transformations. In the context of parameter transformations by parameter we mean *intrinsic parameters*, such as the integral-valued solutions u, v, t of quartic Diophantine equations, $t^2 = Q(u, v)$, with u and v having no common prime factors. The term "intrinsic" is used to convey their characteristic behavior and to distinguish them from the "free" parameters, not subject to any constraints.

We arrive at the concept of intrinsic parameters through "homogenization". E.g. instead of considering the problem of finding the rational points $P(x, y)$ on a quartic (or cubic when $e_0 = 0$) elliptic curve

$$y^2 = e_0x^4 + e_1x^3 + e_2x^2 + e_3x + e_4 \equiv Q(x, 1),$$

we apply the substitutions, $x = u/v$, $y = t/v^2$, and consider the equivalent problem of finding integral-valued solutions of $t^2 = Q(u, v)$. The numerators and denominators, u and v, thus divorced from each other, that is liberated from being formal parts of a fraction, will then behave as independent entities in the course of further calculations. For example, they will transform independently of each other.

Theoretically, all quartic and cubic Diophantine equations, which can also be looked upon as functional equations satisfied by certain elliptic functions, have their own systems of intrinsic parameters. Due to the large variety of functional equations between elliptic functions, the number of possible parameters is also large. For Eq. (1) alone the number of parameters that are more significant is over 50.

When applying a *parameter transformation*, some parameters take on new values while others remain invariant. Examples of ratios (in the case of Eq. (1)) whose values may remain invariant are as follows:

$$I_1 = (C + A)/(B + D), \qquad I_2 = (C^2 + A^2)/(B^2 + D^2),$$
$$I_3 = (C^2 + A^2)/(B^2 - D^2).$$
$$I_4 = (A^2 - B^2 + C^2 - D^2)/(AB - CD),$$
$$I_5 = (A^2 + B^2 + C^2 + D^2)/(AB - CD),$$
$$I_6 = \tau = (B + C)(B + D)/(A^2 - B^2 + C^2 + D^2),$$
$$I_7 = I_4 \cdot (B - D)/(A - B + C + D).$$

The transformations can be characterized and classified according to which of these and other ratios remain invariant.

We shall call a transformation *dual*, if the transformation formulas are symmetric in terms of the old and new values of transforming parameters. Of the presently known transformations only two types are dual, but the methods based on these two supply the largest number of new solutions in relatively small integers.

The *Simple Dual Transformation* (SD) is characterized by having two invariants of type I_1 and four invariants of type I_6. As a method, it is equivalent to the method of Lander, thus in general it results in the same eight new solutions.

The *Composite Dual Transformation* (CD) has $I_6 = \tau$ (or any of the other 15 similar expressions) as its main invariant. A further invariant is I_7 or a similarly built expression. When using it as method, it is more prolific than the SD, since the number of new solutions obtainable from a nontrivial solution is equal to 32. If the method is applied to one of the derived solutions (and this itself is nontrivial), then of the 32 newer solutions one is identical to the solution used at the outset.

Invariably, all methods of solving Eq. (1), which use one known solution to derive another, imply also a simultaneous transformation of parameters. However, when we talk about parameter transformations as methods, we mean *carrying out the computations in terms of the parameters*. This means a reduction of computational efforts, since the parameters are in general numerically smaller than the original variables A, B, C, D or p, q, r, s.

The dual transformations were discovered by the author in the years 1973–1974. Since then they have been used with success, as witnessed by the great number of solutions marked by SD or CD in Table 3. Unfortunately, lack of space does not permit to present here a more detailed account of them.

However, there exist equivalent methods that can readily be defined. For the SD this is Lander's method, already mentioned (see Section 8), or the semisolution method using (11), (12) and discriminant-form (61). The F-solution (30)–(34) provides methods equivalent to SD or to CD. More particularly, when (35) or (36) are used to form chain-generating equations, the results are methods equivalent to SD, and when (37) is used, we have a method equivalent to CD. Accordingly, the initial and the derived solutions, (Solutions 3 and 41), of the numerical example of Section 7 are composite dual transforms of each other.

10. Parametric Solutions. Many numerical solutions can be obtained from formulas of two-parameter solutions. The simplest set of formulas, denoted $E(u, v)$, is the following:

$$A = A(u, v) = u^7 + u^5v^2 - 2u^3v^4 - 3u^2v^5 + uv^6,$$

or, giving only the coefficients,

$$(62) \qquad \begin{aligned} A &= (1,0,1,0,-2,-3,1,0), \\ B &= A(v,-u) = (0,1,3,-2,0,1,0,1), \\ C &= A(u,-v) = (1,0,1,0,-2,3,1,0), \\ D &= A(v,u) = (0,1,-3,-2,0,1,0,1). \end{aligned}$$

These find their origin in Euler [2], but in their present form are due to Gérardin [12].

In contrast to the intrinsic parameters, the parameters u, v of two-parameter formulas are without any constraints. However, to avoid obvious common factors in the values of A, B, C, D, we choose for u and v integers that are relatively prime.

The modern way of computing sets of two-parameter solutions is by applying one of the existing parameter transformation methods to an already known parametric solution. E.g. $E(u, v)$ can be derived by applying the simple dual transformation to the trivial solution $(A, B, C, D) = (u, v, u, -v)$. Similarly, by applying SD to $E(u, v)$, four new sets of parametric solutions are obtained [18], [19]. Including CD in the process will result in further sets of solutions. Of the rich variety of solutions found in this way we cite below the two simplest ones, denoted $P_1(u, v)$ and $P_2(u, v)$. They are of 11th and 13th degree, respectively.

$$(63)\quad \begin{cases} A = (-1,-1,4,17,33,49,58,52,32,12,2,0), \\ B = (1,4,8,7,5,17,44,64,58,34,12,2), \\ C = (1,3,8,13,9,-13,-44,-64,-58,-34,-12,-2), \\ D = (1,2,2,7,27,59,78,66,36,12,2,0). \end{cases}$$

$$(64)\quad \begin{cases} A = (1,3,10,22,44,67,88,95,84,58,30,10,2,0), \\ B = (0,0,3,9,24,45,72,91,94,80,54,28,10,2), \\ C = (1,3,10,22,40,63,82,95,94,80,54,28,10,2), \\ D = (0,2,5,15,28,47,64,73,66,48,26,10,2,0). \end{cases}$$

Some of the simplest numerical solutions are special instances of these parametric solutions. E.g. Solution 3 can be obtained as $P_1(1, 1)$ as well as $P_2(1, 1)$, Solution 2 as $P_1(1, -2)$ and Solution 4 as $P_1(2, -1)$.

Obviously, the number of numerical solutions obtainable from two-parameter formulas and the number of two-parameter formulas themselves is infinite. However, it is not known whether or not every numerical solution of Eq. (1) can be represented as a special case of a parametric solution.

11. The List of Numerical Solutions. In Table 3 we present all known primitive and nontrivial solutions of Eq. (1) whose peaks do not exceed 10^6. Accordingly, the list includes the solutions of the LPS lists, too, these occupying the first 46 entries.

The solutions are listed in their normal forms (see Section 2) and in the increasing order of their peaks. For reference purposes they are numbered with serials 1 through 218. The kth solution in the list will be denoted by S_k.

In the "Remark" column the abbreviations PT (Pythagorean triplets), SD (Simple dual transformation), CD (Composite dual transformation), 2S (Two-solution method, see Section 8), SS (Semi-solution method), refer to particular methods by which the solutions were obtained. The notation FS(...), with a numerical value between the parentheses, refers to F-solutions of the type (41)–(42). The inscribed number gives the value of the invariant τ. $E(u, v)$, $P_1(u, v)$ and $P_2(u, v)$ denote, respectively, solutions computed from sets of two-parameter formulas given by (62), (63) and (64), respectively, with parameter values u and v.

At some of the first 46 solutions the Remark box is left blank, indicating that these solutions would not have been discovered yet without the special computer method producing the LPS lists.

Some interesting finds are also among the solutions. S_{114} has the property that the values of A and B have a common factor greater than 1, namely 41. Accordingly, we have a numerical solution of the equation $41^4 \cdot (a^4 + b^4) = c^4 + d^4$ with values

$a = 1447$, $b = 3271$, $c = 123497$, $d = 100807$. Further, there are three solutions that are linked together by having their common origin in the triple coincidence

$$401168^4 - 17228^4 = 415137^4 - 248289^4 = 421296^4 - 273588^4,$$

or

$$4^4(100292^4 - 4307^4) = 3^4(138379^4 - 82763^4) = 12^4(35108^4 - 22799^4).$$

Keeping one equation at a time and simplifying by possible common factors, we obtain S_{107}, S_{118} and S_{164}. The last solution, S_{164}, was discovered by this observation.

TABLE 3††

A list of primitive nontrivial solutions of the equation $A^4 + B^4 = C^4 + D^4$ in the range $A, B, C, D < 10^6$

NO	A	B	C	D	REMARK
1	134	133	158	59	E(2,1),FS(2),FS(6),FS(-1/4),FS(3/4),PT
2	7	239	227	157	P1(1,-2),FS(2),FS(-4/25),PT
3	256	257	292	193	P1(1,1),P2(1,1),SD OF S2,FS(2), FS(-4/25),PT
4	298	497	502	271	P1(2,-1),FS(-1/4),FS(-6/25),PT
5	514	359	542	103	SS,PT
6	222	631	558	503	SS,FS(-6/25),PT
7	76	1203	1176	653	E(3,1),FS(6),FS(12),FS(-2/9),FS(4/9), PT
8	878	1381	1342	997	PT
9	1324	2189	1784	1997	PT
10	1042	2461	2026	2141	SS,PT
11	248	2797	2524	2131	P2(1,-2),SD OF S3,FS(2),PT
12	1034	2949	2854	1797	SS,PT
13	2986	2345	3190	1577	P2(2,-1),SD OF S4,FS(-1/4),PT
14	2338	3351	3494	1623	E(3,2),FS(3/4)
15	661	3537	3147	2767	PT
16	3364	4849	4288	4303	SD OF S2,FS(2),PT
17	2694	4883	3966	4397	FS(40/9)

†† In the Remark column the notation "S" followed by a number should read with the number in subscript position. Thus e.g. the notation "S22" means "S_{22}", etc.

TABLE 3 (*continued*)

18	604	5053	5048	1283	PT
19	4840	5461	6140	2027	PT
20	274	6619	5942	5093	PT
21	3070	6701	6730	2707	SD OF S4,FS(-1/4),FS(-6/25),PT
22	498	6761	5222	6057	FS(6/25),PT
23	1259	7557	7269	4661	PT
24	6336	7037	7604	5181	SS
25	7432	7559	8912	1657	FS(-4/25),PT
26	6262	8961	7234	8511	SS
27	6842	8409	9018	4903	P1(2,1),FS(3/4),PT
28	5098	9043	6742	8531	P1(2,-3),FS(3/4),PT
29	635	9109	9065	3391	FS(234/25),PT
30	1104	9253	8972	5403	FS(-14/225),PT
31	1142	9289	4946	9097	PT
32	4408	9197	9316	173	
33	5452	9733	7528	9029	CD OF S22,FS(6/25),PT
34	7054	9527	10142	3401	
35	5277	10409	9517	8103	
36	8332	9533	10652	3779	SS
37	3644	11515	5960	11333	FS(-6/49),FS(-66/1225),PT
38	2903	12231	10381	10203	SD OF S1
39	3550	12213	12234	1525	FS(-9/100),PT
40	1149	12653	12167	7809	SD OF S17,FS(40/9)
41	12772	9153	13472	5121	SD OF S19,CD OF S3
42	5526	13751	11022	12169	
43	6470	14421	14190	8171	SS
44	6496	14643	13268	11379	
45	261	14861	14461	8427	SD OF S6,FS(-6/25)
46	581	15109	14723	8461	SD OF S36
47	6101	15265	13085	12743	SD OF S25,FS(-4/25)
48	15594	6485	15642	5675	PT
49	4441	15869	14767	11291	PT
50	7168	16293	15188	11877	FS(10/9),PT
51	691	16377	15663	10411	PT
52	15566	13297	16886	9649	PT

TABLE 3 (*continued*)

53	17236	6673	17332	529	E(4,1)
54	4058	20117	17554	16213	PT
55	4091	22131	21027	14539	SD OF S48,PT
56	21526	19447	23702	14321	FS(171/100),FS(138/289),PT
57	6502	24207	9738	24079	PT
58	19218	25451	27294	5653	FS(-30/289),PT
59	758	27407	27374	7217	SD OF S39,FS(-9/100)
60	15393	27785	25355	22107	FS(40/9),PT
61	2558	28061	28058	4189	FS(56),PT
62	12787	30411	26511	24959	PT
63	5468	31731	25596	27661	SD OF S15
64	6484	32187	29812	23109	E(4,3)
65	4535	32241	32237	5565	SD OF S78
66	7713	36977	34107	26851	SD OF S20,PT
67	13348	37721	37868	167	PT
68	25489	38281	36001	30713	CD OF S3,PT
69	21676	38939	39448	17701	FS(-6/25),PT
70	3080	39789	30348	35885	SD OF S3,PT
71	11888	40465	40540	2513	E(5,1),FS(-4/25)
72	28544	41591	43676	11447	SD OF S9,PT
73	1499	44203	43007	25097	CD OF S3,FS(-4/25),FS(-72/289),PT
74	15052	45453	41324	34419	PT
75	18292	45883	46136	10757	P1(1,-3),FS(6),PT
76	41524	43847	49792	26887	PT
77	31494	53935	35710	52881	E(5,3)
78	45942	55247	53742	48271	PT
79	28997	60369	59777	33237	P1(1,2),FS(6),PT
80	5966	61583	61478	17743	PT
81	38078	60763	62206	29531	P2(2,1),SD OF S28,FS(3/4)
82	23841	64369	60033	46063	SD OF S8
83	61528	45471	65196	27103	FS(6)
84	60328	56941	66308	45869	PT
85	33050	68303	46130	65521	PT
86	3698	72121	70594	38599	PT
87	1661	73059	71807	37143	SD OF S10

TABLE 3 (*continued*)

88	11884	73833	37404	72599	SD OF S36
89	5728	74253	54212	68301	CD OF S8,PT
90	6464	74411	54044	68587	CD OF S89,PT
91	22813	78021	71089	58593	SD OF S31
92	8427	80399	79419	37631	PT
93	14493	81539	80623	37593	CD OF S6,FS(-6/25),PT
94	37996	81885	54520	78621	SD OF S50,FS(10/9),PT
95	23359	83771	74167	66269	PT
96	39393	87797	85173	55073	PT
97	15322	89345	59678	84545	E(5,2)
98	37686	90017	81622	69471	2S FROM S13 AND S15
99	27879	90829	89841	43307	P1(3,-2),FS(-2/9),PT
100	89236	58231	93032	2359	PT
101	37879	94543	92213	55733	SD OF S42
102	17006	97681	29882	97489	PT
103	1788	101819	60752	98427	SD OF S33,FS(6/25),PT
104	47139	103543	98049	72389	PT
105	57832	103809	83004	94529	P1(3,-1),FS(-2,9),PT
106	13614	104909	57582	102451	SD OF S42
107	100292	68397	105324	4307	PT
108	5444	106931	78952	97907	SD OF S89,PT
109	99978	76405	107478	27275	PT
110	29286	117473	111838	76767	SD OF S56,FS(171/100),FS(138/289)
111	12840	126253	72960	122579	PT
112	39717	126659	109213	104133	PT
113	110758	108619	127034	73547	PT
114	59327	134111	123497	100807	PT
115	34813	134413	114613	111637	SD OF S1,FS(2)
116	122664	112507	139356	55483	SD OF S205
117	3800	140047	49328	139505	PT
118	91196	138379	140432	82763	PT
119	125844	135829	143844	113003	PT
120	8052	144401	135504	99409	CD OF S23,PT
121	72274	144733	73766	144541	CD OF S22,PT
122	91508	147941	99848	145627	PT

TABLE 3 (*continued*)

123	78804	153863	129644	133383	PT
124	151394	92839	154522	73703	CD OF S1,SD OF S27,FS(3/4),PT
125	157582	85491	158642	77811	FS(3/4)
126	28580	160133	159544	56635	SD OF S181
127	126168	164705	131760	161951	PT
128	113690	156939	166314	54155	CD OF S50,FS(10/9)
129	125516	161405	174484	7805	E(5,4)
130	29259	175033	156241	136131	PT
131	18657	178559	178509	33499	PT
132	171266	148247	191218	50327	PT
133	48478	198665	168254	166135	FS(-9/100),PT
134	6758	200635	36350	200581	FS(56),PT
135	190444	207971	191512	207139	PT
136	153664	203349	213672	116309	PT
137	219256	47769	219372	23641	SD OF S147,FS(42)
138	88198	226063	138394	219121	CD OF S4,PT
139	22125	228901	228825	44393	PT
140	81416	235201	233212	109951	FS(396/625),PT
141	248034	134611	252974	64851	FS(-560/7569),PT
142	53797	253163	249751	122527	SD OF S10
143	112304	255295	253172	131455	P2(1,2),FS(6),SD OF S75
144	243690	196343	255718	164745	PT
145	32458	261143	88046	260311	PT
146	72489	266063	230099	217443	FS(10/9)
147	266116	52361	266192	36553	FS(42),PT
148	95248	282751	277724	151361	SD OF S37,FS(-6/49),FS(-66/1225)
149	287178	67429	287394	20773	E(6,1)
150	283546	226531	308822	35683	PT
151	166448	331047	295116	208441	PT
152	30519	334883	327183	162869	SD OF S83,FS(6)
153	136321	342081	328619	220803	CD OF S120,PT
154	217863	348197	315957	289111	PT
155	240394	332259	349582	155997	SD OF S8,PT
156	146514	354041	350254	183033	P2(2,-3),SD OF S27,FS(3/4)
157	177070	356307	338310	251501	SD OF S95

TABLE 3 (*continued*)

158	130841	357787	356663	149387	SD OF S5
159	143066	362975	358090	191137	SD OF S35
160	33058	374989	338918	284813	SD OF S5
161	94108	378507	333384	301387	SD OF S2
162	238231	379915	338231	323605	PT
163	379674	157775	382090	96207	P2(1,-3),SD OF S79,FS(6)
164	17228	415137	401168	248289	TRIPLE COINCIDENCE (SEE SEC. 11)
165	31238	419909	419762	81659	E(7,1)
166	19687	421653	410253	239359	PT
167	389242	381583	441718	279311	SD OF S22,FS(6/25),PT
168	292304	454681	335108	439847	SD OF S18
169	348208	476025	396792	450695	PT
170	345588	444311	480032	108201	E(7,3)
171	482944	106163	483172	70157	PT
172	418394	405359	487906	176687	SD OF S73,FS(-4/25),FS(-72/289)
173	485288	378327	500508	338921	SD OF S135
174	59870	515353	175754	513625	CD OF S31,SD OF S102
175	142934	519249	487022	360303	P2(3,-1),SD OF S99,FS(-2/9)
176	452420	434539	525152	196565	CD OF S1,FS(6)
177	149317	533957	498473	376271	PT
178	504474	364829	535658	111459	CD OF S3,SD OF S68
179	119014	539943	470878	435687	SD OF S12
180	490250	500971	548278	417515	CD OF S13,PT
181	258176	547461	554092	57069	PT
182	227697	558305	531145	377271	PT
183	346622	565325	564730	349171	PT
184	21103	569609	569459	102653	SD OF S61,FS(56)
185	50131	571037	570971	86299	CD OF S61,SD OF S134,FS(56)
186	317810	622241	627862	261985	SD OF S6,FS(-6/25)
187	37945	631909	630563	191905	SD OF S9
188	358894	633457	537338	554063	PT
189	34468	634003	278128	628051	PT
190	214349	635423	623861	341849	CD OF S67,PT
191	196179	639311	599511	445397	P1(3,-4),FS(4/9)
192	426592	616049	640612	305713	PT

TABLE 3 (*continued*)

193	3119	641471	567683	505829	FS(2)
194	507934	589471	657848	81249	P1(3,1),FS(4/9)
195	424494	674693	535674	629819	FS(-9/100)
196	14586	683105	635586	483295	E(6,5)
197	558182	711809	590654	694079	CD OF S2,PT
198	651215	727017	720115	660483	PT
199	232484	739885	520640	691859	E(7,5)
200	465236	747633	614656	682161	PT
201	689308	564749	756424	100019	SD OF S16,FS(2)
202	421689	763169	726783	550489	SD OF S4,FS(-6/25)
203	751414	399679	766018	38017	SD OF S49
204	367486	774887	778382	328807	SD OF S48
205	305123	785947	766783	459407	PT
206	16409	826669	804679	467443	SD OF S5
207	532244	827969	768896	869313	PT
208	842204	438241	850912	354271	PT
209	219380	858201	840360	463207	SD OF S2
210	244553	864709	730471	726091	CD OF S13,SD OF S138
211	329626	867849	538734	838711	SD OF S60,FS(40/9)
212	840766	518255	869338	161105	SD OF S2
213	69892	875477	241352	874219	PT
214	3106	884947	400262	875539	E(7,2)
215	505481	905509	874987	623833	SD OF S4
216	897898	465669	906222	387653	SD OF S28,FS(3/4)
217	168824	909613	877004	553453	PT
218	230394	925087	769086	787873	FS(-60/361),PT

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